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A basic difficulty of the 'local mean-field' theory in a plasma

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Abstract. An exact formulation of the framework of the so-called 'local mean-field' theory is given in a plasma with an arbitrary number of components and it is proved that an approximate theory with a frequency-independent 'local-field' correction factor $G_{ij}(\boldsymbol{q})$ has a basic difficulty. The difficulty is as follows. When $G_{ij}(\boldsymbol{q})$ is chosen so as to satisfy the third frequency moment sum rule, the theory has the unphysical result: (a) $1/\epsilon(q, 0)$ does not vanish as $q \to 0$ in an electrically neutral system without a uniform, neutralising background charge and (b) $\chi_s = 0$ in a magnetically neutral system, $\epsilon(q, \omega)$ and χ_s being the generalised dielectric function and the spin susceptibility, respectively. An approximation procedure for removing this difficulty is discussed.

1. Introduction

A response of a plasma to a weak external field is described by a response function. Transforming the response function appropriately, we can describe exactly the response in such a picture that the system responds to an effective field as if it were an assembly of free particles distributed in momentum space according to a real-particle momentum distribution function, where the effective field is the sum of the Hartree field and the so-called 'local-field' correction (see Nozieres and Pines 1958a, b). In this picture the response function is a functional of the real-particle momentum distribution function and the so-called 'local-field' correction factor (LFCF).

In the random phase approximation (RPA) the effective field and the real-particle momentum distribution function are approximated by the Hartree field and the non-interacting-particle momentum distribution function, respectively. Since RPA was considered to be adequate only in the case of the weak coupling, how to proceed beyond RPA has become one of the problems of main concern for theoretical physicists, and various 'local mean-field' theories (LFT) have been proposed by Vashishta and Singwi (1972), Vashishta *et al* (1974), and many other workers. On the other hand, some structure of the LFCF has been also reported; Goodman and Sjölander (1973) and Sjölander (1974) found that in a uniform electron liquid with a positive background LFT with a frequency-independent LFCF necessarily violates either the spin-susceptibility sum rule or the third sum rule, and one of the present authors (Ginoza 1977) proved that in a two-component plasma such a LFT never satisfies the perfect screening sum rule and the third sum rule simultaneously.

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Now, the electron liquid may be regarded as a 'two-component' system magnetically; namely, spin-up and spin-down electrons. Though the discussion by Goodman and Sjölander and that by one of the present authors are concerned with the magnetic and electric responses, respectively, the latter is similar to the former in the sense that each discussion considers the two-component system and is closely related to a singular term in the third frequency moment of the response function. In fact, the singular term is typical for the response function of the two-component system. Furthermore, according to the physics behind the term investigated by Goodman and Sjölander, such a term may be characteristic of a response function of an arbitrary multi-component system. Therefore, these two apparently separate discussions may be treated from a unified point of view and further generalised to a single discussion of a plasma with an arbitrary number of components. The purpose of the present paper is to investigate the discussion from such a viewpoint. A discussion on removing the difficulty above will also be added.

The electric- and magnetic-response functions may be described from the unified point of view by introducing the concept of *subcomponent* which will be defined in the next section. It will be proved that, when the LFCF is chosen so as both to be frequency-independent and to satisfy the third frequency moment in the system which is neutral both electrically and magnetically as a whole, the choice involves such an unphysical result as

$$\lim_{q\to 0} 1/\epsilon(q, 0) \neq 0 \tag{1.1}$$

and

$$\chi_s = 0, \tag{1.2}$$

where $\epsilon(q, \omega)$ and χ_s are the generalised dielectric function and the spin susceptibility, respectively. In a sense, this result may have already been suggested and our treatment may be somewhat mathematical. We would like to point out, however, that equation (1.1) is not true in the system with a uniform neutralising background charge even if it is a multi-component system. Applying the result to a two-subcomponent plasma to which the electron liquid with a uniform positive background and the two-component plasma consisting of spinless particles belong, we reach directly the discussions of Goodman and Sjölander and one of the present authors.

In § 2, the concept of subcomponent will be defined and the expressions for frequency moments of a response function will be presented. The exact formulation of the framework of LFT will be given in § 3. Equations (1.1) and (1.2) will be proved in § 4. In § 5, an approximation procedure for removing the unphysical result described above will be discussed.

2. Frequency moments

Let us consider a uniform, non-relativistic quantum plasma which is contained in a large box of unit volume with periodic boundary condition and is electrically neutral as a whole. We will regard only Coulomb interaction as the interaction between particles, and hence the z-component of spin-magnetic moment is conserved in the process of the interaction. We will use this quantity for specification of particles. This is useful because, as is seen in the following, it enables us to discuss the electric- and the spin magnetic-response functions from a unified point of view. Let the assembly of particles specified by mass m_i, charge e_i, and the z-component of spin-magnetic moment μ_i be called the *i-subcomponent.* The system under consideration then has an arbitrary number of subcomponents, each being specified by an integer $1, 2, \ldots, n$ and i-subcomponent consisting of n_i particles.

Now, let us denote the destruction and creation operators for a particle with momentum $\boldsymbol{k}$ belonging to the i-subcomponent by $C_i(\boldsymbol{k})$ and $C_i^\dagger(\boldsymbol{k})$, respectively. The $\boldsymbol{q}$th charge-density fluctuation operator of i-subcomponent is

$$\rho_i(\boldsymbol{q}) = e_i \sum_{\boldsymbol{k}} C_i^\dagger(\boldsymbol{k}-\boldsymbol{q}/2)C_i(\boldsymbol{k}+\boldsymbol{q}/2).$$

Let $V_i^{\text{ext}}(\boldsymbol{q}, \omega)$ be the amplitude of the potential of an external field with wavevector $\boldsymbol{q}$ and frequency ω which would couple only with the i-subcomponent. The linear response of $\rho_i(\boldsymbol{q})$ to such fields is

$$\rho_i^{\text{ind}}(\boldsymbol{q}, \omega) = \sum_j D_{ij}^{\text{r}}(\boldsymbol{q}, \omega) V_j^{\text{ext}}(\boldsymbol{q}, \omega), \tag{2.1}$$

where $D_{ij}^{\text{r}}(\boldsymbol{q}, \omega)$ is a retarded response function, dependent on subcomponent, defined as

$$D_{ij}^{\text{r}}(\boldsymbol{q}, \omega) = -\text{i}\int_0^\infty \text{d}t \langle[\rho_i(\boldsymbol{q}, t), \rho_j^\dagger(\boldsymbol{q})]\rangle\, \text{e}^{\text{i}\omega t}. \tag{2.2}$$

Here $\langle\ldots\rangle$ means canonical ensemble average and $\rho_i(\boldsymbol{q}, t)$ is the Heisenberg representation of $\rho_i(\boldsymbol{q})$. Knowledge of this function permits us to obtain various response functions of the system. In particular, a generalised dielectric function $\epsilon(q, \omega)$ and a spin magnetic response function $\chi^{\text{r}}(q, \omega)$ are obtained from

$$1/\epsilon(q, \omega) = 1 + v(q)\sum_i\sum_j D_{ij}^{\text{r}}(\boldsymbol{q}, \omega), \tag{2.3a}$$

$$\chi^{\text{r}}(q, \omega) = \sum_i\sum_j \lambda_i\lambda_j D_{ij}^{\text{r}}(\boldsymbol{q}, \omega), \tag{2.3b}$$

with $v(q) = (4\pi/q^2)(1-\delta_{\boldsymbol{q}0})$ and $\lambda_i = \mu_i/e_i$. The spin susceptibility is then

$$\chi_{\text{s}} = -\lim_{q\to 0} \chi^{\text{r}}(q, 0). \tag{2.4}$$

Integrating equation (2.2) by parts successively and using the Heisenberg equation of motion, we obtain

$$v(q)D_{ij}^{\text{r}}(\boldsymbol{q}, \omega) = \sum_{l=1} M_{l,ij}(\boldsymbol{q})/\omega^{l+1}, \tag{2.5}$$

where

$$M_{l,ij}(\boldsymbol{q}) = v(q)\langle[[\rho_i(\boldsymbol{q}), H]_l, \rho_j^\dagger(\boldsymbol{q})]\rangle,$$

$[\rho_i(\boldsymbol{q}), H]_l$ being the lth order commutator of $\rho_i(\boldsymbol{q})$ with the Hamiltonian H. When we calculate the first three frequency moments, we have

$$M_{1,ij}(\boldsymbol{q}) = \delta_{ij}\omega_i^2, \tag{2.6a}$$

$$M_{2,ij}(\boldsymbol{q}) = 0, \tag{2.6b}$$

$$M_{3,ij}(\boldsymbol{q}) = \delta_{ij}M_{3,i}^{(0)}(q) + \omega_i^2\omega_j^2\Big(1 + \Gamma_{ij} + \sum_{\boldsymbol{q}'} (\boldsymbol{q}\boldsymbol{q}'/qq')^2 \times[(1-\delta_{\boldsymbol{q}\boldsymbol{q}'})g_{ij}(\boldsymbol{q}'-\boldsymbol{q}) - (1-\delta_{0\boldsymbol{q}'})g_{ij}(\boldsymbol{q}')]\Big). \tag{2.6c}$$

In the above equations

$$\omega_i^2 = 4\pi n_i e_i^2/m_i,$$

$$M_{3,i}^{(0)}(q) = (8\pi K_i e_i^2/m_i^2)q^2 + (\pi n_i e_i^2/m_i^3)q^4,$$

$$\Gamma_{ij} = \sum_{i'} (\delta_{i'i} - \delta_{ij})(n_{i'}e_{i'}/n_i e_i)[g_{i'j}(r=0) - 1]/3,$$

$$g_{ij}(\boldsymbol{q}) = (\langle \rho_i(\boldsymbol{q})\rho_j^\dagger(\boldsymbol{q})\rangle - n_i e_i^2 \delta_{ij})/(n_i e_i n_j e_j)$$

and

$$g_{ij}(r) = 1 + \sum_{\boldsymbol{q}\neq 0} g_{ij}(\boldsymbol{q})\,\mathrm{e}^{\mathrm{i}\boldsymbol{qr}},$$

K_i being the averaged kinetic energy of the i-subcomponent. Note that the term with Γ_{ij} in equation (2.6*c*) exists only in the multi-subcomponent system and that it becomes dominant in the small wavenumber region. The existence of this singular term introduces a basic difficulty into LFT as is proved in § 4.

3. Exact formulation of LFT

Let us define a Green function as follows:

$$R_{i;j}(\boldsymbol{k};\boldsymbol{q},t) = -\mathrm{i}\theta(t)\langle[C_i^\dagger(\boldsymbol{k}-\boldsymbol{q}/2,t)C_i(\boldsymbol{k}+\boldsymbol{q}/2,t), \rho_j^\dagger(\boldsymbol{q})]\rangle.$$

Making use of an equation of motion for this function, we obtain an equation for D_{ij}^{r} as

$$D_{ij}^{\mathrm{r}}(\boldsymbol{q},\omega) = \delta_{ij}D_i^{(0)}(\boldsymbol{q},\omega) + v(q)D_i^{(0)}(\boldsymbol{q},\omega)\sum_{i'} D_{i'j}^{\mathrm{r}}(\boldsymbol{q},\omega) + R_{ij}(\boldsymbol{q},\omega), \tag{3.1}$$

where

$$D_i^{(0)}(\boldsymbol{q},\omega) = e_i^2 \sum_{\boldsymbol{k}} \frac{n_i(|\boldsymbol{k}-\boldsymbol{q}/2|) - n_i(|\boldsymbol{k}+\boldsymbol{q}/2|)}{\omega - \boldsymbol{kq}/m_i + \mathrm{i}0^+},$$

$$n_i(k) = \langle C_i^\dagger(\boldsymbol{k})C_i(\boldsymbol{k})\rangle,$$

$$R_{ij}(\boldsymbol{q},\omega) = \sum_{i'} e_i^2 e_{i'} \sum_{\boldsymbol{q}'} v(q')\sum_{\boldsymbol{k}}\sum_{\boldsymbol{k}'}[F_i(\boldsymbol{k}+\boldsymbol{q}'/2;\boldsymbol{q},\omega)$$
$$- F_i(\boldsymbol{k}-\boldsymbol{q}'/2;\boldsymbol{q},\omega)]R_{ii';j}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q}-\boldsymbol{q}',\boldsymbol{q}',\omega),$$

$$F_i(\boldsymbol{k};\boldsymbol{q},\omega) = (\omega - \boldsymbol{kq}/m_i + \mathrm{i}0^+)^{-1},$$

$$R_{ii';j}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q},\boldsymbol{q}',\omega) = -\mathrm{i}\int_0^\infty \mathrm{d}t\langle[\hat{\gamma}_{ii'}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q},\boldsymbol{q}',t), \rho_j^\dagger(\boldsymbol{q}+\boldsymbol{q}')]\rangle\,\mathrm{e}^{\mathrm{i}\omega t}, \tag{3.1a}$$

and $\hat{\gamma}_{ii'}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q},\boldsymbol{q}',t)$ is the Heisenberg representation of an operator defined as

$$\hat{\gamma}_{ii'}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q},\boldsymbol{q}') = C_i^\dagger(\boldsymbol{k}-\boldsymbol{q}/2)C_{i'}^\dagger(\boldsymbol{k}'-\boldsymbol{q}'/2)C_{i'}(\boldsymbol{k}'+\boldsymbol{q}'/2)C_i(\boldsymbol{k}+\boldsymbol{q}/2)$$
$$- \delta_{\boldsymbol{q}0}n_i(k)C_{i'}^\dagger(\boldsymbol{k}'-\boldsymbol{q}'/2)C_{i'}(\boldsymbol{k}'+\boldsymbol{q}'/2) - \delta_{\boldsymbol{q}'0}n_{i'}(k')C_i^\dagger(\boldsymbol{k}-\boldsymbol{q}/2)C_i(\boldsymbol{k}+\boldsymbol{q}/2).$$

In order to proceed beyond RPA, we must determine $R_{ij}(\boldsymbol{q},\omega)$. In this paper, however, we are interested in the structure of an effective field. Therefore we shall transform $R_{ij}(\boldsymbol{q},\omega)$ into a new function $G_{ij}(\boldsymbol{q},\omega)$ defined as

$$G_{ij}(\boldsymbol{q},\omega) = -(\hat{R}\hat{D}^{-1})_{ij}/(vD_i^{(0)}). \tag{3.2}$$

In this definition, $\hat{R}$ is a matrix whose (i, j) element is $R_{ij}(\boldsymbol{q}, \omega)$, while $\hat{D}^{-1}$ is the inverse of a matrix $\hat{D}$ whose (i, j) element is $D^{\mathrm{r}}_{ij}(\boldsymbol{q}, \omega)$.

Let us investigate the physical content of this newly introduced function. Inverting equation (2.1) we get

$$V_i^{\mathrm{ext}}(\boldsymbol{q}, \omega) = \sum_j (\hat{D}^{-1})_{ij} \rho_j^{\mathrm{ind}}(\boldsymbol{q}, \omega). \tag{3.3}$$

Multiplying both sides of equation (3.1) by $V_j^{\mathrm{ext}}(\boldsymbol{q}, \omega)$, summing over j on both sides, and using equations (2.1), (3.2), and (3.3), we have

$$\rho_i^{\mathrm{ind}}(\boldsymbol{q}, \omega) = D_i^{(0)}(\boldsymbol{q}, \omega) V_i^{\mathrm{eff}}(\boldsymbol{q}, \omega),$$

where

$$V_i^{\mathrm{eff}}(\boldsymbol{q}, \omega) = V_i^{\mathrm{ext}}(\boldsymbol{q}, \omega) + v(q) \sum_j \rho_j^{\mathrm{ind}}(\boldsymbol{q}, \omega) - v(q) \sum_j G_{ij}(\boldsymbol{q}, \omega) \rho_j^{\mathrm{ind}}(\boldsymbol{q}, \omega).$$

This result shows that the i-subcomponent responds to the effective external field given by V_i^{eff} as if it were an assembly of free particles distributed in momentum space according to $n_i(k)$. The effective field consists of the Hartree field, which is the sum of the first and second terms, and the field resulting from correlation effects. The function $G_{ij}(\boldsymbol{q}, \omega)$ is thus a factor determining the latter field. Such a picture for the response mechanism of the system has given a useful framework for the treatment of the correlation effects. A theory based on this picture may be given by a self-consistent choice of $n_i(k)$ and $G_{ij}(\boldsymbol{q}, \omega)$. We will, conventionally, call such a theory 'local mean-field' theory (LFT), though a field given by G_{ij} may not be local in general.

Eliminating $R_{ij}(\boldsymbol{q}, \omega)$ from equations (3.1) and (3.2), we obtain

$$(\hat{D}^{-1})_{ij} = vA_{ij},$$

where

$$A_{ij}(\boldsymbol{q}, \omega) = G_{ij}(\boldsymbol{q}, \omega) - 1 + \delta_{ij}/[v(q) D_i^{(0)}(\boldsymbol{q}, \omega)].$$

The expression of D^{r}_{ij} then becomes

$$D^{\mathrm{r}}_{ij} = \Delta_{ji}/[v \det(A_{ij})], \tag{3.4}$$

where Δ_{ji} is the cofactor of A_{ji} in a matrix $\hat{A}$ whose (i, j) element is A_{ij}. This allows us to obtain expressions for various response functions in terms of $A_{ij}(\boldsymbol{q}, \omega)$; namely $G_{ij}(\boldsymbol{q}, \omega)$ and $n_i(k)$. In particular, the expression for $\epsilon(q, \omega)$ is obtained by substituting equation (3.4) into equation (2.3a) and using the identity $\Sigma_i \Sigma_j \Delta_{ij} = \det(A_{ij} + 1) - \det(A_{ij})$ as

$$1/\epsilon(q, \omega) = \det(A_{ij} + 1)/\det(A_{ij}). \tag{3.5}$$

Now, the high-frequency expansion of equation (3.4) gives

$$v(q) D^{\mathrm{r}}_{ij}(\boldsymbol{q}, \omega) = \delta_{ij} \omega_i^2 \Big(\sum_{\boldsymbol{k}} n_i(k)/n_i \Big) \Big/ \omega^2 + \{M_{3,i}^{(0)}(q) \delta_{ij} + \omega_i^2 \omega_j^2 [1 - G_{ij}(\boldsymbol{q}, \infty)]\}/\omega^4 + \dots.$$

This must be equal to equation (2.5) because equation (3.4) is not an approximation at all. We thus obtain

$$\sum_{\boldsymbol{k}} n_i(k) = n_i, \tag{3.6}$$

$$G_{ij}(\boldsymbol{q}, \infty) = F_{ij}/Q_i + \sum_{\boldsymbol{q}'} (\boldsymbol{q}\boldsymbol{q}'/qq')^2 [(1 - \delta_{\boldsymbol{q}'0}) g_{ij}(\boldsymbol{q}') - (1 - \delta_{\boldsymbol{q}'\boldsymbol{q}}) g_{ij}(\boldsymbol{q}' - \boldsymbol{q})], \tag{3.7}$$

where $Q_i = n_i e_i$ and

$$F_{ij} = \sum_{i'} Q_{i'}(\delta_{ii'} - \delta_{ij})[1 - g_{i'j}(r=0)]/3. \tag{3.8}$$

These equations give some constraints on the choice of $n_i(k)$ and $G_{ij}(\boldsymbol{q}, \omega)$. Equation (3.6) is a conservation law of particle number, while equation (3.7) gives the exact expression of $G_{ij}(\boldsymbol{q}, \omega)$ in the high-frequency limit. These are also necessary and sufficient conditions in order that $D^{\mathrm{r}}_{ij}(\boldsymbol{q}, \omega)$ given by a particular choice of $G_{ij}(\boldsymbol{q}, \omega)$ and $n_i(k)$, assumed to be analytic in the upper half of the complex ω-plane, satisfies the f-sum rule and third sum rule.

4. The proof of the basic difficulty of a LFT

Let us assume a theory in which LFCF does not depend on ω, and let us denote this by $G_{ij}(\boldsymbol{q})$. If this theory satisfies the third sum rule, then $G_{ij}(\boldsymbol{q})$ must be equal to equation (3.7). As is proved below, however, this choice involves the unphysical result like equations (1.1) and (1.2). Now, since the second term on the right-hand side of equation (3.7) and $[v(q)D^{(0)}_i(\boldsymbol{q}, 0)]^{-1}$ are of the order of q^2 as $q \to 0$, the above choice of G_{ij} yields

$$\lim_{q\to 0} A_{ij}(\boldsymbol{q}, 0) = F_{ij}/Q_i - 1 + \mathrm{O}(q^2). \tag{4.1}$$

The proof of equation (1.1). From equations (3.5) and (4.1), we obtain

$$\lim_{q\to 0} 1/\epsilon(q, 0) = [\det(F_{ij}/Q_i) + \mathrm{O}(q^2)]/[\det(F_{ij}/Q_i - 1) + \mathrm{O}(q^2)].$$

Since

$$\sum_i F_{ij} = 0, \tag{4.2}$$

as is seen from equation (3.8), we have

$$\det(F_{ij}/Q_i) = \left(1\Big/\prod_{i=1}^{n} Q_i\right) \det(F_{ij})$$

$$= \left(1\Big/\prod_{i=1}^{n} Q_i\right) \begin{vmatrix} \sum_i F_{i1} & \sum_i F_{i2} & \cdots & \sum_i F_{in} \\ F_{21} & F_{22} & \cdots & F_{2n} \\ \vdots & \vdots & & \vdots \\ F_{n1} & F_{n2} & \cdots & F_{nn} \end{vmatrix} = 0.$$

Similarly, we can show with the use of equation (4.2) that

$$\det(F_{ij}/Q_i - 1) = \left(1\Big/\prod_{i=1}^{n} Q_i\right) C \sum_i Q_i \tag{4.3}$$

where

$$C = \begin{vmatrix} -1 & -1 & \cdots & -1 \\ F_{21} - Q_2 & F_{22} - Q_2 & \cdots & F_{2n} - Q_2 \\ \vdots & \vdots & & \vdots \\ F_{n1} - Q_n & F_{n2} - Q_n & \cdots & F_{nn} - Q_n \end{vmatrix}.$$

Therefore

$$\lim_{q\to 0} 1/\epsilon(q,0) = \mathrm{O}(q^2)\Big/\Big[C\sum_i Q_i + \mathrm{O}(q^2)\Big]$$

and equation (1.1) is proved.

The proof of equation (1.2). The spin susceptibility χ_s is given by equation (2.4). By substituting equation (3.4) into equation (2.3*b*) and multiplying the numerator and the denominator of this equation by $\Pi_{i=1}^{n} Q_i$, we get

$$v(q)\chi^{\mathrm{r}}(q,\omega) = \sum_{i=1} \lambda_i Q_i F_i/\det(Q_i A_{ij}),$$

where

$$F_i = \begin{vmatrix} Q_1A_{11} & \cdots & Q_1A_{1n} \\ \vdots & & \vdots \\ Q_{i-1}A_{i-11} & \cdots & Q_{i-1}A_{i-1n} \\ \lambda_1 & \cdots & \lambda_n \\ Q_{i+1}A_{i+11} & \cdots & Q_{i+1}A_{i+1n} \\ \vdots & & \vdots \\ Q_nA_{n1} & \cdots & Q_nA_{nn} \end{vmatrix}.$$

Now, $\Sigma_i \lambda_i Q_i F_i = \Sigma_i \lambda_i Q_i F_1 + \Sigma_i \lambda_i Q_i (F_i - F_1) = \Sigma_i \lambda_i Q_i (F_i - F_1)$ by virtue of $\Sigma_i \mu_i n_i = 0$ (magnetic neutrality of the system as a whole). Then

$$v(q)\chi^{\mathrm{r}}(q,\omega) = \sum_i \lambda_i Q_i \begin{vmatrix} Q_1A_{11}+Q_iA_{i1} & \cdots & Q_1A_{1n}+Q_iA_{in} \\ Q_2A_{21} & \cdots & Q_2A_{2n} \\ \vdots & & \vdots \\ Q_{i-1}A_{i-11} & \cdots & Q_{i-1}A_{i-1n} \\ \lambda_1 & \cdots & \lambda_n \\ Q_{i+1}A_{i+11} & \cdots & Q_{i+1}A_{i+1n} \\ \vdots & & \vdots \\ Q_nA_{n1} & \cdots & Q_nA_{nn} \end{vmatrix} \Big/\det(Q_iA_{ij})$$

$$= \sum_i \lambda_i \begin{vmatrix} \sum_j Q_jA_{j1} & \cdots & \sum_j Q_jA_{jn} \\ A_{21} & \cdots & A_{2n} \\ \vdots & & \vdots \\ A_{i-11} & \cdots & A_{i-1n} \\ \lambda_1 & \cdots & \lambda_n \\ A_{i+11} & \cdots & A_{i+1n} \\ \vdots & & \vdots \\ A_{n1} & \cdots & A_{nn} \end{vmatrix} \Big/[Q_1 \det(A_{ij})].$$

Substituting equation (4.1) into this equation for $q \to 0$ and using equations (4.2) and (4.3), we obtain from equation (2.4)

$$\chi_s = -(q^2/4\pi)\sum_i \lambda_i Q_i\Big[C_i\sum_j Q_j + \mathrm{O}(q^2)\Big]\Big/\Big[C\sum_j Q_j + \mathrm{O}(q^2)\Big],$$

where

$$C_i = \begin{vmatrix} -1 & \cdots & -1 \\ F_{21} - Q_2 & \cdots & F_{2n} - Q_2 \\ \vdots & & \vdots \\ F_{i-11} - Q_{i-1} & \cdots & F_{i-1n} - Q_{i-1} \\ \lambda_1 & \cdots & \lambda_n \\ F_{i+11} - Q_{i+1} & \cdots & F_{i+1n} - Q_{i+1} \\ \vdots & & \vdots \\ F_{n1} - Q_n & \cdots & F_{nn} - Q_n \end{vmatrix}.$$

Therefore equation (1.2) is proved.

5. Discussion

In the previous section, we have proved equations (1.1) and (1.2) as the basic difficulty of the theory whose LFCF does not depend on frequency ω. This difficulty means that the ω-dependence of the LFCF is essential for the proper description of both low- and high-ω phenomena. Let us discuss how to take this ω-dependence into account correctly.

In order to obtain an idea on the correct consideration of the ω-dependence of the LFCF, we investigate in some detail the origin of this difficulty in the case of the electron liquid with a uniform positive background charge. As is seen from the proofs in the previous section, the origin of the difficulty can be traced to the singular term of equation (2.6c). Such a term remains in the third frequency moment of $\chi^{\mathrm{r}}(q, \omega)$, but not in that of $D^{\mathrm{r}}(q, \omega)$. The difficulty, corresponding to this, is only the magnetic one like equation (1.2). Note, however, that the disappearance of the electric difficulty like equation (1.1) is due to the rigidity of the background charge. Now, Goodman and Sjölander (1973) investigated whether the singular term of $\chi^{\mathrm{r}}(q, \omega)$ is closely related to the response of multi-pair excitations. On the other hand, Kalia and Mukhopadhyay (1974) pointed out the discrepancy of the theory of Vashishta and Singwi (1972) for $D^{\mathrm{r}}(q, \omega)$ from the scattering experiment data in the position and width of the peak of a dynamical structure factor. This may suggest the importance of the damping mechanism other than Landau damping. The multi-pair excitations can result in such a damping mechanism. Therefore, the origins of the difficulties of the theories for $\chi^{\mathrm{r}}(q, \omega)$ and $D^{\mathrm{r}}(q, \omega)$ described above, where these functions can be given by the same $G_{\sigma\sigma'}$ from a unified viewpoint, may be closely related to each other and the correct consideration of the effect of the multi-pair excitations may shed light on the solution to this problem. This leads us to taking the dynamics of two-particle correlation into account correctly.

Now, equation (3.1a) describes the response of the correlation between the ($\boldsymbol{ik}$)-particle and the ($i'\boldsymbol{k}'$)-particle to an external field and its equation of motion may govern the dynamics of two-particle correlation. When we attempt to solve this equation, the problem of how to treat the term involving the Coulomb interaction

Hamiltonian H_1 may arise. Here, we shall take the following approximation:

$$[\hat{\gamma}_{ii'}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q},\boldsymbol{q}'),H_1]=(\tfrac{1}{2})\sum_{\boldsymbol{q}''}v(q'')\{[\hat{\gamma}_{ii'}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q},\boldsymbol{q}'),\rho(-\boldsymbol{q}'')]\rho(\boldsymbol{q}'')$$
$$+\rho(\boldsymbol{q}'')[\hat{\gamma}_{ii'}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q},\boldsymbol{q}'),\rho(-\boldsymbol{q}'')]\}$$
$$\doteqdot v(|\boldsymbol{q}+\boldsymbol{q}'|)\langle[\hat{\gamma}_{ii'}(\boldsymbol{k},\boldsymbol{k}';\boldsymbol{q},\boldsymbol{q}'),\rho(-\boldsymbol{q}-\boldsymbol{q}')]\rangle\rho(\boldsymbol{q}+\boldsymbol{q}'),$$

where $\rho(\boldsymbol{q})=\Sigma_i\,\rho_i(\boldsymbol{q})$. The result obtained with this approximation has properties as follows:

(a) the ω-dependence of LFCF is taken into account so that the first and third frequency moment sum rules are satisfied and the unphysical result like equations (1.1) and (1.2) does not arise;

(b) the expression of LFCF obtained is exact for sufficiently large q or ω;

(c) from the result for sufficiently large q we can obtain the well-known relation

$$[\partial g_{ij}(r)/\partial r]_{r=0}=g_{ij}(r=0)/a_{ij'}$$

where $a_{ij}=(1/m_i+1/m_j)/(2e_ie_j)$;

(d) the damping mechanism other than Landau damping is taken into account.

Therefore, the result may have the correct ω-dependence of the LFCF.

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